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Thermal Relaxation of a QED Cavity

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Abstract We study repeated interactions of the quantized electromagnetic field in a cavity with single two-levels atoms. Using the Markovian nature of the resulting quantum evolution we study its large time asymptotics. We show that, whenever the atoms are distributed according to the canonical ensemble at temperature $T > 0$ and some generic non-degeneracy condition is satisfied, the cavity field relaxes towards some invariant state. Under some more stringent non-resonance condition, this invariant state is thermal equilibrium at some renormalized temperature T^*. Our result is non-perturbative in the strength of the atom-field coupling. The relaxation process is slow (non-exponential) due to the presence of infinitely many metastable states of the cavity field.

Keywords Thermal relaxation · Completely positive maps · One-atom maser · QED cavity · Repeated interactions · Rabi oscillations

1 Introduction

Open Systems During the last years there has been a growing interest for the rigorous development of the quantum statistical mechanics of open systems. Such a system consists in a confined subsystem $\mathcal{S}$ in contact with an environment made of one or several extended subsystems $\mathcal{R}_1, \ldots$ usually called reservoirs. We refer the reader to [9] and in particular to the review article [4] for a modern introduction to the subject.

Dedicated to Jürg Fröhlich and Thomas Spencer on the occasion of their 60th birthday.

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Two different approaches have been used to study open systems: Hamiltonian and Markovian. The first one is fundamental. It is based on a complete description of the microscopic dynamics of the coupled system $\mathcal{S}+\mathcal{R}_1+\cdots$. One uses traditional tools of quantum mechanics—spectral analysis and scattering theory – to study this dynamics. So far, most results obtained in this way are perturbative in the system-reservoir coupling and, for technical reasons, limited to small systems $\mathcal{S}$ described by a finite dimensional Hilbert space (e.g., N-level atoms).

In the Markovian approach, one gives up the microscopic description of the reservoirs and tries to describe directly the effective dynamics of the "small" system $\mathcal{S}$ under the influence of its environment. This evolution is governed by a quantum master equation which defines a semi-group of completely positive, trace preserving maps on the state space of $\mathcal{S}$ (see Definition 4.1 below). There are two ways to justify such a Markovian dynamics: as a scaling limit of the microscopic dynamics of the coupled system $\mathcal{S}+\mathcal{R}_1+\cdots$ (e.g., the van Hove weak coupling limit [20, 21, 23, 25]), or as the result of driving the system $\mathcal{S}$ with stochastic forces (quantum Langevin equation [31]).

Equilibrium vs. Nonequilibrium When the environment is in thermal equilibrium, the basic problem is thermal relaxation: does the small subsystem $\mathcal{S}$ return to a state of thermal equilibrium? In the cases when $\mathcal{S}$ has a finite dimensional Hilbert space and the environment consists of an ideal quantum gas, this question has been extensively investigated in [10, 24, 28, 32].

Open systems become more interesting when their environment is not in thermal equilibrium. Suppose for example that $\mathcal{S}$ is brought into contact with several reservoirs, each of them being in a thermal equilibrium state but with different intensive thermodynamic parameters. Then one expects the joint system $\mathcal{S}+\mathcal{R}_1+\cdots$ to relax towards a non-equilibrium steady states (NESS). Such states have been constructed in [2, 3, 18, 19, 33, 41, 44, 47]. They carry currents, have non vanishing entropy production rate, ... These transport properties were investigated in [5, 17, 29, 43]. The linear response theory (Green-Kubo formula, Onsager reciprocity relations, central limit theorem) was developed in [29, 34–38]. Moreover, current fluctuations and related problems (Evans-Searles and Gallavotti-Cohen symmetries) were studied in [22, 26, 50].

Repeated Interactions Motivated by several new physical applications as well as by their attractive mathematical structure, a class of open systems has recently become very popular in the literature: repeated interaction (RI) systems. There, the environment consists in a sequence $\mathcal{E}_1, \mathcal{E}_2, \ldots$ of independent subsystems. The "small" subsystem $\mathcal{S}$ interacts with $\mathcal{E}_1$ during the time interval $[0, \tau_1[$, then with $\mathcal{E}_2$ during the interval $[\tau_1, \tau_1+\tau_2[$, etc. While $\mathcal{S}$ interacts with $\mathcal{E}_m$, the other elements of the sequence evolve freely according to their intrinsic (uncoupled) dynamics. Thus, the evolution of the joint system $\mathcal{S}+\mathcal{E}_1+\cdots$ is completely determined by the sequence $\tau_1, \tau_2, \ldots$, the individual dynamics of each $\mathcal{E}_m$ and the coupled dynamics of each pair $\mathcal{S}+\mathcal{E}_m$.

In the simplest RI models each $\mathcal{E}_m$ is a copy of some $\mathcal{E}$, $\tau_m \equiv \tau$, and the dynamics of $\mathcal{E}_m$ and $\mathcal{S}+\mathcal{E}_m$ are independent of m, generated by some Hamiltonians $H_{\mathcal{E}}$, $H_{\mathcal{S}\mathcal{E}}$. Such models have been analyzed in [14, 55] (see also [15] for a random setting). It was shown in [14] that the RI dynamics gives rise to a Markovian effective dynamics on the system $\mathcal{S}$ and drives the latter to an asymptotic state, at an exponential rate (provided $\mathcal{S}$ has a finite dimensional Hilbert space). The limit $\tau \to 0$ with appropriate rescaling of the interaction Hamiltonian $H_{\mathcal{S}\mathcal{E}}$ was studied in [7, 8]. In this scaling limit, RI systems become continuous interaction systems and the effective dynamics on $\mathcal{S}$ converges towards a continuous semigroup of completely positive maps associated with a quantum Langevin equation. Related results pertaining to various other scaling limits of RI systems have also been investigated in [6] with similar results.

Due to their particular structure, RI systems are both Hamiltonian (with a time-dependent Hamiltonian) and Markovian (the effective dynamics of $\mathcal{S}$ is described by a discrete semi-group of completely positive maps, see Sect. 2.2 for the precise meaning of this statement). For that reason, we believe that these models provide a useful framework to develop our understanding of various aspects of the quantum statistical mechanics of open systems.

In the physical paradigm of a RI system, $\mathcal{S}$ is the quantized electromagnetic field of a cavity through which a beam of atoms, the $\mathcal{E}_m$, is shot in such a way that no more than *one atom* is present in the cavity at any time. Such systems play a fundamental role in the experimental and theoretical investigations of basic matter-radiation processes. They are also of practical importance in quantum optics and quantum state engineering [42, 46, 53–55]. So-called "One-Atom Masers", where the beam is tuned in such a way that at each given moment a single atom is inside a microwave cavity and the interaction time τ is the same for each atom, have been experimentally realized in laboratories [42, 54].

In this paper we start the mathematical analysis of a specific model of RI system describing the one-atom maser experiment mentioned above (a precise description of the model is given in Sect. 2). We consider here the first natural question, namely that of thermal relaxation: is it possible to thermalize a mode of a QED cavity by means of 2-level atoms if the latter are initially at thermal equilibrium? The non-equilibrium situation (NESS, entropy production, fluctuation symmetries) will be considered in [13]. We would like to emphasize that in our situation the Hilbert space of the small system $\mathcal{S}$ is *not* finite dimensional. Moreover, we do not make use of any perturbation theory, i.e., our results do not restrict to small coupling constants.

The paper is organized as follows: The precise description of the model is given in Sect. 2 and the main results are stated and discussed in Sect. 3. Proofs will be found in Sect. 4.

2 Description of the Model

2.1 The Jaynes–Cummings Atom–Field Dynamics

We consider the situation where atoms of the beam are prepared in a stationary mixture of two states with energies $E_0 < E_1$ and we assume the cavity to be nearly resonant with the transitions between these two states. Neglecting the non-resonant modes of the cavity, we can describe its quantized electromagnetic field by a single harmonic oscillator of frequency $\omega \simeq \omega_0 \equiv E_1 - E_0$.

The Hilbert space for a single atom is $\mathcal{H}_{\mathcal{E}} \equiv \mathbb{C}^2$ which, for notational convenience, we identify with $\Gamma_-(\mathbb{C})$, the Fermionic Fock space over $\mathbb{C}$. Without loss of generality we set $E_0 = 0$. The Hamiltonian of a single atom is thus

$$H_{\mathcal{E}} \equiv \omega_0 b^* b,$$

where b^*, b denote the creation/annihilation operators on $\mathcal{H}_{\mathcal{E}}$. Stationary states of the atom can be parametrized by the inverse temperature $\beta \in \mathbb{R}$ and are given by the density matrices $\rho_{\mathcal{E}}^{\beta} \equiv \mathrm{e}^{-\beta H_{\mathcal{E}}}/\mathrm{Tr}\,\mathrm{e}^{-\beta H_{\mathcal{E}}}$.

The Hilbert space of the cavity field is $\mathcal{H}_{\mathcal{S}} \equiv \ell^2(\mathbb{N}) = \Gamma_+(\mathbb{C})$, the Bosonic Fock space over $\mathbb{C}$. Its Hamiltonian is

$$H_{\mathcal{S}} \equiv \omega N \equiv \omega a^* a,$$

where a^*, a are the creation/annihilation operators on $\mathcal{H}_\mathcal{S}$ satisfying the commutation relation $[a, a^*] = I$. Normal states of $\mathcal{S}$ are density matrices, positive trace class operators ρ on $\mathcal{H}_\mathcal{S}$ with $\operatorname{Tr}\rho = 1$. We will use the notation $\rho(A) \equiv \operatorname{Tr}(\rho A)$ for $A \in \mathcal{B}(\mathcal{H}_\mathcal{S})$. These are the only states we shall consider on $\mathcal{S}$. Therefore, in the following, "state" always means "normal state" or equivalently "density matrix". Moreover, we will say that a state is diagonal if it is represented by a diagonal matrix in the eigenbasis of $H_\mathcal{S}$.

In the dipole approximation, an atom interacts with the cavity field through its electric dipole moment. The full dipole coupling is given by $(\lambda/2)(a + a^*) \otimes (b + b^*)$, acting on $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$, where $\lambda \in \mathbb{R}$ is a coupling constant. Neglecting the counter rotating term $a \otimes b + a^* \otimes b^*$ in this coupling (this is the so called *rotating wave approximation*) leads to the well known Jaynes-Cummings Hamiltonian

$$H \equiv H_\mathcal{S} \otimes \mathbb{1}_\mathcal{E} + \mathbb{1}_\mathcal{S} \otimes H_\mathcal{E} + \lambda V, \quad V \equiv \frac{1}{2}(a^* \otimes b + a \otimes b^*), \tag{2.1}$$

for the coupled system $\mathcal{S} + \mathcal{E}$ (see e.g., [11, 16, 27]). The operator H has a distinguished property which allows for its explicit diagonalisation: it commutes with the total number operator

$$M \equiv a^* a + b^* b. \tag{2.2}$$

An essential feature of the dynamics generated by H are Rabi oscillations. In the presence of n photons, the probability for the atom to make a transition from its ground state to its excited state is a periodic function of time. The circular frequency of this oscillation is given by $\sqrt{\lambda^2 n + (\omega_0 - \omega)^2}$, a fact easily derived from the propagator formula (4.2) below. Thus, in our units, λ is the one photon Rabi-frequency of the atom in a perfectly tuned cavity.

The rotating wave approximation, and thus the dynamics generated by the Jaynes-Cummings Hamiltonian, is known to be in good agreement with experimental datas as long as the detuning parameter $\Delta \equiv \omega - \omega_0$ satisfies $|\Delta| \ll \min(\omega_0, \omega)$ and the coupling is small $|\lambda| \ll \omega_0$. However, we are not aware of any mathematically precise statement about this approximation.

2.2 Repeated Interaction Dynamics

Given an interaction time $\tau > 0$, the system $\mathcal{S}$ successively interacts with different copies of the system $\mathcal{E}$, each interaction having a duration τ. The issue is to understand the asymptotic behavior of the system $\mathcal{S}$ when the number of such interactions tends to $+\infty$ (which is equivalent to time t going to $+\infty$). The Hilbert space describing the entire system $\mathcal{S} + \mathcal{C}$ then writes

$$\mathcal{H} \equiv \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{C}, \quad \mathcal{H}_\mathcal{C} \equiv \bigotimes_{n \geq 1} \mathcal{H}_{\mathcal{E}_n},$$

where $\mathcal{H}_{\mathcal{E}_n}$ are identical copies of $\mathcal{H}_\mathcal{E}$. During the time interval $[(n-1)\tau, n\tau)$, the system $\mathcal{S}$ interacts only with the n-th element of the chain. The evolution is thus described by the Hamiltonian H_n which acts as H on $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_{\mathcal{E}_n}$ and as the identity on the other factors $\mathcal{H}_{\mathcal{E}_k}$.

Remark A priori we should also include the free evolution of the non-interacting elements of $\mathcal{C}$. However, since we shall take the various elements of $\mathcal{C}$ to be initially in thermal equilibrium, this free evolution will not play any role.

Given any initial state ρ on $\mathcal{S}$ and assuming that all the atoms are in the stationary state $\rho_{\mathcal{E}}^{\beta}$, the state of the total repeated interaction system after n interactions is thus given by

$$\mathrm{e}^{-i\tau H_n}\cdots\mathrm{e}^{-i\tau H_1}\Big(\rho\otimes\bigotimes_{k\geq 1}\rho_{\mathcal{E}}^{\beta}\Big)\mathrm{e}^{i\tau H_1}\cdots\mathrm{e}^{i\tau H_n}.$$

To obtain the state ρ_n of the system $\mathcal{S}$ after these n interactions we take the partial trace over the chain $\mathcal{C}$, i.e.,

$$\rho_n=\mathrm{Tr}_{\mathcal{H}_{\mathcal{C}}}\Big[\mathrm{e}^{-i\tau H_n}\cdots\mathrm{e}^{-i\tau H_1}\Big(\rho\otimes\bigotimes_{k\geq 1}\rho_{\mathcal{E}}^{\beta}\Big)\mathrm{e}^{i\tau H_1}\cdots\mathrm{e}^{i\tau H_n}\Big].\tag{2.3}$$

It is easy to make sense of this formal expression (we deal here with countable tensor products). Indeed, at time $n\tau$ only the n first elements of the chain have played a role so that we can replace $\bigotimes_{k\geq 1}\rho_{\mathcal{E}}^{\beta}$ by $\rho_{\mathcal{E}}^{\beta\,(n)}\equiv\bigotimes_{k=1}^{n}\rho_{\mathcal{E}}^{\beta}$ and the partial trace over the chain by the partial trace over the finite tensor product $\mathcal{H}_{\mathcal{C}}^{(n)}\equiv\bigotimes_{k=1}^{n}\mathcal{H}_{\mathcal{E}_k}$.

The very particular structure of the repeated interaction systems allows us to rewrite ρ_n in a much more convenient way. The two main characteristics of these repeated interaction systems are:

1. The various elements of $\mathcal{C}$ do not interact directly (only via the system $\mathcal{S}$),
2. The system $\mathcal{S}$ interacts only once with each element of $\mathcal{C}$, and with only one at any time.

It is therefore easy to see that the evolution of the system $\mathcal{S}$ is Markovian: the state ρ_n only depends on the state ρ_{n-1} and the n-th interaction. More precisely, one can write (see also [6, 14])

$$\begin{aligned}\rho_n&=\mathrm{Tr}_{\mathcal{H}_{\mathcal{C}}^{(n)}}\big[\mathrm{e}^{-i\tau H_n}\cdots\mathrm{e}^{-i\tau H_1}\big(\rho\otimes\rho_{\mathcal{E}}^{\beta\,(n)}\big)\mathrm{e}^{i\tau H_1}\cdots\mathrm{e}^{i\tau H_n}\big]\\&=\mathrm{Tr}_{\mathcal{H}_{\mathcal{E}_n}}\big[\mathrm{e}^{-i\tau H_n}\big(\mathrm{Tr}_{\mathcal{H}_{\mathcal{C}}^{(n-1)}}\big[\mathrm{e}^{-i\tau H_{n-1}}\cdots\mathrm{e}^{-i\tau H_1}\big(\rho\otimes\rho_{\mathcal{E}}^{\beta\,(n-1)}\big)\mathrm{e}^{i\tau H_1}\cdots\mathrm{e}^{i\tau H_{n-1}}\big]\otimes\rho_{\mathcal{E}}^{\beta}\big)\mathrm{e}^{i\tau H_n}\big]\\&=\mathrm{Tr}_{\mathcal{H}_{\mathcal{E}_n}}\big[\mathrm{e}^{-i\tau H_n}\big(\rho_{n-1}\otimes\rho_{\mathcal{E}}^{\beta}\big)\mathrm{e}^{i\tau H_n}\big],\end{aligned}$$

that is

$$\rho_n=\mathcal{L}_{\beta}(\rho_{n-1}),$$

with

$$\mathcal{L}_{\beta}(\rho)\equiv\mathrm{Tr}_{\mathcal{H}_{\mathcal{E}}}\big[\mathrm{e}^{-i\tau H}\big(\rho\otimes\rho_{\mathcal{E}}^{\beta}\big)\mathrm{e}^{i\tau H}\big].\tag{2.4}$$

Definition 2.1 The map $\mathcal{L}_{\beta}$ defined on the set $\mathcal{J}_1(\mathcal{H}_{\mathcal{S}})$ of trace class operators on $\mathcal{H}_{\mathcal{S}}$ by (2.4) is called the reduced dynamics. The state of $\mathcal{S}$ evolves according to the discrete semigroup $\{\mathcal{L}_{\beta}^{n}\,|\,n\in\mathbb{N}\}$ generated by this map:

$$\rho_n=\mathcal{L}_{\beta}^{n}(\rho).$$

In particular, a state ρ is invariant iff $\mathcal{L}_{\beta}(\rho)=\rho$.

Note that $\mathcal{L}_{\beta}$ is clearly a contraction. To understand the asymptotic behavior of ρ_n, we shall study its spectral properties. In particular, we will be interested in its peripheral eigenvalues $\mathrm{e}^{i\theta}$, for $\theta\in\mathbb{R}$.

Remark When the atom-field coupling is turned off, the reduced dynamics is nothing but the free evolution of $\mathcal{S}$, i.e., $\mathcal{L}_\beta(\rho) = \mathrm{e}^{-i\tau H_\mathcal{S}}\rho\mathrm{e}^{i\tau H_\mathcal{S}}$. Note that $\mathcal{J}_1(\mathcal{H}_\mathcal{S}) = \bigoplus_{d\in\mathbb{Z}} \mathcal{J}_1^{(d)}(\mathcal{H}_\mathcal{S})$ where each subspace

$$\mathcal{J}_1^{(d)}(\mathcal{H}_\mathcal{S}) \equiv \{X \in \mathcal{J}_1(\mathcal{H}_\mathcal{S}) \,|\, \mathrm{e}^{-i\theta N} X \mathrm{e}^{i\theta N} = \mathrm{e}^{i\theta d} X \text{ for all } \theta \in \mathbb{R}\}, \tag{2.5}$$

is infinite dimensional (it is the set of bounded operators X which, in the canonical basis of $\mathcal{H}_\mathcal{S} = \ell^2(\mathbb{N})$, have a matrix representation $X_{nm} = x_n\,\delta_{n+d,m}$ with $\sum_{n\geq 0} |x_n| < \infty$). Thus, for $\lambda = 0$, the spectrum of $\mathcal{L}_\beta$ is pure point

$$\mathrm{sp}(\mathcal{L}_\beta) = \mathrm{sp}_{\mathrm{pp}}(\mathcal{L}_\beta) = \{\mathrm{e}^{i\tau\omega d} \,|\, d \in \mathbb{Z}\}.$$

This spectrum is finite if $\tau\omega \in 2\pi\mathbb{Q}$ and densely fills the unit circle in the opposite case. In both cases, all the eigenvalues (and in particular 1) are infinitely degenerate. This explains why perturbation theory in λ fails for this model.

3 Results

To formulate our main results we need a notion of Rabi resonance. Such a resonance occurs when the interaction time τ is an integer multiple of the period of a Rabi oscillation. Here and in the following we will use the dimensionless detuning parameter and coupling constant

$$\eta \equiv \left(\frac{\Delta\tau}{2\pi}\right)^2, \qquad \xi \equiv \left(\frac{\lambda\tau}{2\pi}\right)^2,$$

to parametrize our model.

Definition 3.1 Let n be a positive integer. We shall say that n is a Rabi resonance if

$$\xi n + \eta = k^2, \tag{3.1}$$

for some positive integer k and denote by $R(\eta, \xi)$ the set of Rabi resonances.

The following elementary lemma (see Sect. 4.10 for a discussion) shows that, depending on η and ξ, the system has either no, one or infinitely many Rabi resonances. We shall say accordingly that it is non-resonant, simply resonant or fully resonant. A fully resonant system will be called degenerate if there exist $n \in \{0\} \cup R(\eta, \xi)$ and $m \in R(\eta, \xi)$ such that $n < m$ and $n+1, m+1 \in R(\eta, \xi)$.

Lemma 3.2 1. *If η and ξ are both irrational then the system can be either non-resonant or simply resonant.*

2. *If one of them is rational and the other not, then the system is non-resonant.*

3. *If they are both rational, write their irreducible representations as $\eta = a/b$, $\xi = c/d$, denote by m the least common multiple of b and d and set*

$$\mathfrak{X} \equiv \{x \in \{0, \ldots, \xi m - 1\} \,|\, x^2 m \simeq \eta m (\mathrm{mod}\, \xi m)\}.$$

The system is non-resonant if $\mathfrak{X}$ is empty. In the opposite case it is fully resonant and

$$R(\eta, \xi) = \{(k^2 - \eta)/\xi \,|\, k = jm\xi + x,\, j \in \mathbb{N}, x \in \mathfrak{X}\} \cap \mathbb{N}^*.$$

4. *A necessary condition for the system to be degenerate is that both ξ and η be integers such that $\eta > 0$ is a quadratic residue modulo ξ, i.e., there exists an integer y such that $\eta = y^2$ modulo ξ.*

The Hilbert space $\mathcal{H}_{\mathcal{S}}$ has a decomposition

$$\mathcal{H}_S = \bigoplus_{k=1}^{r} \mathcal{H}_{\mathcal{S}}^{(k)}, \tag{3.2}$$

where $r-1$ is the number of Rabi resonances, $\mathcal{H}_{\mathcal{S}}^{(k)} \equiv \ell^2(I_k)$ and $\{I_k \,|\, k = 1, \dots, r\}$ is the partition of $\mathbb{N}$ induced by the resonances. More precisely we set

$$\begin{array}{ll} I_1 \equiv \mathbb{N} & \text{if } R(\eta,\xi) \text{ is empty,} \\ I_1 \equiv \{0,\dots,n_1-1\},\ I_2 \equiv \{n_1, n_1+1,\dots\} & \text{if } R(\eta,\xi) = \{n_1\}, \\ I_1 \equiv \{0,\dots,n_1-1\},\ I_2 \equiv \{n_1,\dots,n_2-1\},\dots & \text{if } R(\eta,\xi) = \{n_1,n_2,\dots\}. \end{array}$$

We shall say that $\mathcal{H}_{\mathcal{S}}^{(k)}$ is the k-th Rabi sector, denote by P_k the corresponding orthogonal projection and set $l_k \equiv \dim \mathcal{H}_{\mathcal{S}}^{(k)}$.

Thermal relaxation is an ergodic property of the map $\mathcal{L}_\beta$ and of its invariant states. For any density matrix ρ, we denote the orthogonal projection on the closure of $\operatorname{Ran}\rho$ by $s(\rho)$, the support of ρ. We also write $\mu \ll \rho$ whenever $s(\mu) \le s(\rho)$.

A state ρ is ergodic (respectively mixing) for the semigroup generated by $\mathcal{L}_\beta$ whenever

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\mathcal{L}_\beta^n(\mu)\right)(A) = \rho(A), \tag{3.3}$$

(respectively)

$$\lim_{n\to\infty} \left(\mathcal{L}_\beta^n(\mu)\right)(A) = \rho(A), \tag{3.4}$$

holds for all states $\mu \ll \rho$ and all $A \in \mathcal{B}(\mathcal{H}_{\mathcal{S}})$. ρ is exponentially mixing if the convergence in (3.4) is exponential, i.e., if

$$\left|\left(\mathcal{L}_\beta^n(\mu)\right)(A) - \rho(A)\right| \le C_{A,\mu}\, \mathrm{e}^{-\alpha n},$$

for some constant $C_{A,\mu}$ which may depend on A and μ and some $\alpha > 0$ independent of A and μ. A mixing state is ergodic and an ergodic state is clearly invariant.

A state ρ is faithful iff $\rho > 0$, that is $s(\rho) = I$. Thus, if ρ is a faithful ergodic (resp. mixing) state the convergence (3.3) (resp. (3.4)) holds for every state μ and one has global relaxation. In this case, ρ is easily seen to be the only ergodic state of $\mathcal{L}_\beta$. Conversely, one can show (see Theorem 4.4) that if $\mathcal{L}_\beta$ has a unique faithful invariant state, this state is ergodic.

We need some notations to formulate our main result. For $\beta \in \mathbb{R}$ we set $\beta^* \equiv \beta\omega_0/\omega$ and to each Rabi sector $\mathcal{H}_{\mathcal{S}}^{(k)}$ we associate the state

$$\rho_{\mathcal{S}}^{(k)\,\beta^*} \equiv \frac{\mathrm{e}^{-\beta^* H_{\mathcal{S}}} P_k}{\operatorname{Tr} \mathrm{e}^{-\beta^* H_{\mathcal{S}}} P_k} = \frac{\mathrm{e}^{-\beta\omega_0 N} P_k}{\operatorname{Tr} \mathrm{e}^{-\beta\omega_0 N} P_k}.$$

Theorem 3.3 1. *If the system is non-resonant then* $\mathcal{L}_\beta$ *has no invariant state for* $\beta \le 0$ *and the unique ergodic state*

$$\rho_{\mathcal{S}}^{\beta^*} = \frac{\mathrm{e}^{-\beta^* H_{\mathcal{S}}}}{\mathrm{Tr}\,\mathrm{e}^{-\beta^* H_{\mathcal{S}}}}$$

for $\beta > 0$. *In the latter case any initial state relaxes in the mean to the thermal equilibrium state at inverse temperature* β^*.

2. *If the system is simply resonant then* $\mathcal{L}_\beta$ *has the unique ergodic state* $\rho_{\mathcal{S}}^{(1)\,\beta^*}$ *if* $\beta \le 0$ *and two ergodic states* $\rho_{\mathcal{S}}^{(1)\,\beta^*}$, $\rho_{\mathcal{S}}^{(2)\,\beta^*}$ *if* $\beta > 0$. *In the latter case, for any state* μ, *one has*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\mathcal{L}_\beta^n(\mu)\right)(A) = \mu(P_1)\,\rho_{\mathcal{S}}^{(1)\,\beta^*}(A) + \mu(P_2)\,\rho_{\mathcal{S}}^{(2)\,\beta^*}(A), \tag{3.5}$$

for all $A \in \mathcal{B}(\mathcal{H}_{\mathcal{S}})$.

3. *If the system is fully resonant then for any* $\beta \in \mathbb{R}$, $\mathcal{L}_\beta$ *has infinitely many ergodic states* $\rho_{\mathcal{S}}^{(k)\,\beta^*}$, $k = 1, 2, \ldots$. *Moreover, if the system is non-degenerate,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\mathcal{L}_\beta^n(\mu)\right)(A) = \sum_{k=1}^{\infty} \mu(P_k)\,\rho_{\mathcal{S}}^{(k)\,\beta^*}(A), \tag{3.6}$$

holds for any state μ *and all* $A \in \mathcal{B}(\mathcal{H}_{\mathcal{S}})$.

4. *If the system is non-degenerate, any invariant state is diagonal and can be represented as a convex linear combination of ergodic states, i.e., the set of invariant states is a simplex.*

Remarks 1. Notice the renormalization $\beta \to \beta^*$ of the equilibrium temperature when the detuning parameter η in non-zero.

2. In the non-degenerate cases, our result implies some weak form of decoherence in the energy eigenbasis of the cavity field: the time averaged off-diagonal part of the state $\mathcal{L}_\beta^n(\mu)$ decays with time.

3. Assertion 4 shows in particular that in the non-degenerate cases an ergodic decomposition theorem holds. Note that, in contrast with classical dynamical systems, this is not necessarily the case for quantum systems.

4. If the system is degenerate, (3.6) and the conclusions of Assertion 4 still hold provided a further non-resonance condition is satisfied. Namely, we will show that there is a finite non-empty set $\mathcal{D} \subset \mathbb{N}^*$ such that the peripheral eigenvalues of $\mathcal{L}_\beta$ with non-diagonal eigenvectors are given by $\mathrm{e}^{i(\tau\omega+\xi\pi)d}$, $d \in \mathcal{D}$ (see Lemma 4.6 below for details). If $\mathrm{e}^{i(\tau\omega+\xi\pi)d} \neq 1$ for all $d \in \mathcal{D}$, none of these eigenvalue equals 1 and all eigenvectors of $\mathcal{L}_\beta$ to the eigenvalue 1 are diagonal.

The following result brings some additional information on the relaxation process in finite dimensional Rabi sectors.

Theorem 3.4 *Whenever the state* $\rho_{\mathcal{S}}^{(k)\,\beta^*}$ *is ergodic it is also exponentially mixing if the sector* $\mathcal{H}_{\mathcal{S}}^{(k)}$ *is finite dimensional.*

Remark Numerical experiments support the conjecture that all the ergodic states are mixing. However, our analysis does not provide a proof of this conjecture if $\mathcal{H}_{\mathcal{S}}^{(k)}$ is infinite dimensional. In fact, we will see in Sect. 4.5 that $\mathcal{L}_\beta$ has an infinite number of metastable states in the non-resonant and simply resonant cases. As a result, we expect slow (i.e., non-exponential) relaxation (see Paragraph 4.5.4 for illustrations).

4 Proofs

4.1 Preliminaries

The map $\mathcal{L}_\beta$ acts on the set of density matrices on $\mathcal{H}_\mathcal{S}$, but its definition (2.4) obviously extends to the space $\mathcal{J}_1(\mathcal{H}_\mathcal{S})$ of trace class operators on $\mathcal{H}_\mathcal{S}$. Let us first recall some definitions and important results concerning linear maps on trace ideals (we refer to [40, 48, 49] for detailed expositions).

Definition 4.1 Let $\phi : \mathcal{J}_1(\mathcal{H}) \to \mathcal{J}_1(\mathcal{H})$ be a linear map.

1. ϕ is positive if it leaves the cone $\mathcal{J}_{1+}(\mathcal{H})$ of positive trace class operators invariant.
2. ϕ is n-positive if the extended maps $\phi \otimes I$ acting on $\mathcal{J}_1(\mathcal{H}) \otimes \mathcal{B}(\mathbb{C}^n)$ is positive.
3. ϕ is completely positive (CP) if it is n-positive for all $n \in \mathbb{N}$.
4. ϕ is trace preserving if $\mathrm{Tr}(\phi(\rho)) = \mathrm{Tr}(\rho)$ for any $\rho \in \mathcal{J}_1(\mathcal{H})$.

Given a linear map ϕ on $\mathcal{J}_1(\mathcal{H})$, we denote by $r(\phi)$ its spectral radius $\sup\{|z| \mid z \in \mathrm{sp}(\phi)\}$ which, by a result of Gelfand [30], is equal to $\lim_{n\to\infty} \|\phi^n\|^{1/n}$.

Theorem 4.2 *Let ϕ be a positive map on $\mathcal{J}_1(\mathcal{H})$.*

1. *ϕ is bounded.*
2. *If ϕ is CP there exists an at most countable family $(V_i)_{i\in J}$ of bounded operators on $\mathcal{H}$ such that*

$$0 \le \sum_{i\in J'} V_i^* V_i \le I,$$

for any finite $J' \subset J$ and

$$\phi(\rho) = \sum_{i\in J} V_i \rho V_i^*, \tag{4.1}$$

for any $\rho \in \mathcal{J}_1(\mathcal{H})$.

3. *If ϕ is CP and trace preserving then $r(\phi) = \|\phi\| = 1$.*

A decomposition (4.1) of a CP map is called a Kraus representation. Such a representation is not necessarily unique.

The following result due to Schrader ([48], Theorem 4.1) is our main tool for the spectral analysis of $\mathcal{L}_\beta$.

Theorem 4.3 *Let ϕ be a 2-positive map on $\mathcal{J}_1(\mathcal{H})$ such that $r(\phi) = \|\phi\|$. If λ is a peripheral eigenvalue of ϕ with eigenvector X, i.e., $\phi(X) = \lambda X$, $X \neq 0$, $|\lambda| = r(\phi)$, then $|X|$ is an eigenvector of ϕ to the eigenvalue $r(\phi)$: $\phi(|X|) = r(\phi)|X|$.*

Finally, the following theorem reduces the problem of thermal relaxation "in the mean" (in the sense of (3.3)) to the existence and uniqueness of a faithful invariant state.

Theorem 4.4 *Let ϕ be a CP trace preserving map on $\mathcal{J}_1(\mathcal{H})$. If ϕ has a faithful invariant state ρ_{stat} and 1 is a simple eigenvalue of ϕ then ρ_{stat} is ergodic.*

This result is most probably known, at least for strongly continuous semigroups of CP trace preserving maps. Since we are not aware of any reference in the discrete case we provide a proof in Sect. 4.9.

4.2 Strategy

Using Theorem 4.2, the following proposition follows directly from the definition (2.4) of $\mathcal{L}_\beta$.

Proposition 4.5 *$\mathcal{L}_\beta$ is a completely positive, trace preserving map on $\mathcal{J}_1(\mathcal{H}_\mathcal{S})$. In particular one has $r(\mathcal{L}_\beta) = \|\mathcal{L}_\beta\| = 1$.*

In order to prove Theorems 3.3 and 3.4 we will derive an explicit Kraus representation of $\mathcal{L}_\beta$ in Sect. 4.3. In Sect. 4.4 we will show that $\mathcal{L}_\beta$ leaves the subspaces $\mathcal{J}_1^{(d)}(\mathcal{H}_\mathcal{S})$ invariant. Using the Kraus representation of $\mathcal{L}_\beta$ we will then derive a convenient formula for its action on the subspace $\mathcal{J}_1^{(0)}(\mathcal{H}_\mathcal{S})$ of diagonal matrices. With this formula we will construct all diagonal invariant states in Sect. 4.5. Investigating the block structure of $\mathcal{L}_\beta$ associated to Rabi sectors (Sect. 4.6) will allow us to invoke Theorem 4.3 in Sect. 4.7. In this way we reduce the peripheral eigenvalue problem $\mathcal{L}_\beta(X) = \mathrm{e}^{i\theta}X$, $\theta \in \mathbb{R}$, to diagonal matrices. In Sect. 4.8 the result of this analysis will allow us to conclude the proof.

4.3 Kraus Representation of $\mathcal{L}_\beta$

Denote by $|-\rangle$ and $|+\rangle$ the ground state and the excited state of the atom $\mathcal{E}$. This orthonormal basis of $\mathcal{H}_\mathcal{E}$ allows us to identify $\mathcal{H} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$ with $\mathcal{H}_\mathcal{S} \oplus \mathcal{H}_\mathcal{S}$. Using the fact that H commutes with the total number operator M (recall (2.2)), an elementary calculation shows that, in this representation, the unitary group $\mathrm{e}^{-i\tau H}$ is given by

$$\mathrm{e}^{-i\tau H} = \begin{pmatrix} \mathrm{e}^{-i(\tau\omega N+\pi\eta^{1/2})}\,C(N) & -i\mathrm{e}^{-i(\tau\omega N+\pi\eta^{1/2})}S(N)\,a^* \\ -i\mathrm{e}^{-i(\tau\omega(N+1)+\pi\eta^{1/2})}S(N+1)\,a & \mathrm{e}^{-i(\tau\omega(N+1)+\pi\eta^{1/2})}\,C(N+1)^* \end{pmatrix}, \tag{4.2}$$

where

$$C(N) \equiv \cos(\pi\sqrt{\xi N+\eta}) + i\eta^{1/2}\,\frac{\sin(\pi\sqrt{\xi N+\eta})}{\sqrt{\xi N+\eta}},$$

$$S(N) \equiv \xi^{1/2}\,\frac{\sin(\pi\sqrt{\xi N+\eta})}{\sqrt{\xi N+\eta}},$$

with the convention $\sin(0)/0 = 1$ to avoid any ambiguity in the case $\eta = 0$. Let $w_\beta(\sigma) \equiv \langle\sigma|\rho_\mathcal{E}^\beta|\sigma\rangle = (1+\mathrm{e}^{\sigma\beta\omega_0})^{-1}$ denote the Gibbs distribution of the atoms. The defining identity (2.4) yields

$$\mathcal{L}_\beta(\rho) = \sum_{\sigma,\sigma'}\langle\sigma'|\mathrm{e}^{-i\tau H}|\sigma\rangle w_\beta(\sigma)\rho\langle\sigma|\mathrm{e}^{i\tau H}|\sigma'\rangle = \sum_{\sigma,\sigma'} V_{\sigma'\sigma}\rho V_{\sigma'\sigma}^*, \tag{4.3}$$

where the operators $V_{\sigma'\sigma}$ are given by

$$\begin{aligned} V_{--} &= w_\beta(-)^{1/2}\,\mathrm{e}^{-i\tau\omega N}\,C(N), & V_{-+} &= w_\beta(+)^{1/2}\,\mathrm{e}^{-i\tau\omega N}\,S(N)\,a^*, \\ V_{+-} &= w_\beta(-)^{1/2}\,\mathrm{e}^{-i\tau\omega N}\,S(N+1)\,a, & V_{++} &= w_\beta(+)^{1/2}\,\mathrm{e}^{-i\tau\omega N}\,C(N+1)^*. \end{aligned} \tag{4.4}$$

The above formulas give us an explicit Kraus representation of the CP map $\mathcal{L}_\beta$.

4.4 Action of $\mathcal{L}_\beta$ on Diagonal States

Using the facts that $[H, M] = [H_{\mathcal{E}}, \rho_{\mathcal{E}}^{\beta}] = 0$, one easily shows from the definition (2.4) that

$$\mathcal{L}_\beta(\mathrm{e}^{-i\theta N} X \mathrm{e}^{i\theta N}) = \mathrm{e}^{-i\theta N}\mathcal{L}_\beta(X)\mathrm{e}^{i\theta N},$$

holds for any $X \in \mathcal{J}_1(\mathcal{H}_{\mathcal{S}})$ and $\theta \in \mathbb{R}$. This is of course also evident from the above Kraus representation of $\mathcal{L}_\beta$. However, it is not clear there what properties of the system are responsible for this invariance. It follows that $\mathcal{L}_\beta$ leaves the subspaces $\mathcal{J}_1^{(d)}(\mathcal{H}_{\mathcal{S}})$ (see (2.5)) invariant, and hence admits a decomposition

$$\mathcal{L}_\beta = \bigoplus_{d\in\mathbb{Z}} \mathcal{L}_\beta^{(d)}. \tag{4.5}$$

We shall be particularly interested in the action of $\mathcal{L}_\beta$ on diagonal matrices, i.e., in $\mathcal{L}_\beta^{(0)}$. To understand why, note that if $\rho \in \mathcal{J}_1(\mathcal{H}_{\mathcal{S}})$ is an invariant state then $\rho \geq 0$, $\mathrm{Tr}(\rho) = 1$ and $\mathcal{L}_\beta(\rho) = \rho$. It follows from (4.5) that its diagonal part $\rho^{(0)} \in \mathcal{J}_1^{(0)}(\mathcal{H}_{\mathcal{S}})$ satisfies $\rho^{(0)} \geq 0$, $\mathrm{Tr}(\rho^{(0)}) = 1$ and $\mathcal{L}_\beta^{(0)}(\rho^{(0)}) = \rho^{(0)}$, i.e., $\rho^{(0)}$ is also an invariant state. The problem of existence of an invariant state therefore completely reduces to the existence of the eigenvalue 1 of $\mathcal{L}_\beta^{(0)}$.

Denoting by x_n the diagonal elements of $X \in \mathcal{J}_1^{(0)}(\mathcal{H}_{\mathcal{S}})$, we can identify $\mathcal{J}_1^{(0)}(\mathcal{H}_{\mathcal{S}})$ with $\ell^1(\mathbb{N})$. The Kraus representation derived in the previous subsection immediately yields

$$\begin{aligned}(\mathcal{L}_\beta^{(0)} x)_n = \frac{1}{1+\mathrm{e}^{-\beta\omega_0}} \Bigg[&\Big(\cos^2(\pi\sqrt{\xi n + \eta}) + \mathrm{e}^{-\beta\omega_0}\cos^2(\pi\sqrt{\xi(n+1)+\eta})\Big) x_n \\ &+ \frac{\sin^2(\pi\sqrt{\xi n+\eta})}{\xi n + \eta}\left(\eta x_n + \mathrm{e}^{-\beta\omega_0}\xi n x_{n-1}\right) \\ &+ \frac{\sin^2(\pi\sqrt{\xi(n+1)+\eta})}{\xi(n+1)+\eta}\left(\mathrm{e}^{-\beta\omega_0}\eta x_n + \xi(n+1)x_{n+1}\right)\Bigg].\end{aligned}$$

To rewrite this expression in a more convenient form let us introduce the number operator

$$(Nx)_n \equiv n x_n,$$

as well as the finite difference operators

$$(\nabla x)_n \equiv \begin{cases} x_0 & \text{for } n = 0, \\ x_n - x_{n-1} & \text{for } n \geq 1, \end{cases} \qquad (\nabla^* x)_n \equiv x_n - x_{n+1} \quad (\text{for } n \geq 0),$$

on $\ell^1(\mathbb{N})$. A simple algebra then leads to

$$\mathcal{L}_\beta^{(0)} = I - \nabla^* D(N)\mathrm{e}^{-\beta\omega_0 N}\nabla \mathrm{e}^{\beta\omega_0 N}, \tag{4.6}$$

where

$$D(N) \equiv \frac{1}{1+\mathrm{e}^{-\beta\omega_0}} \sin^2(\pi\sqrt{\xi N + \eta})\,\frac{\xi N}{\xi N + \eta}. \tag{4.7}$$

4.5 Diagonal Invariant States

We are now in position to determine all the diagonal invariant states and more generally all eigenvectors of $\mathcal{L}_\beta^{(0)}$ to the eigenvalue 1. Setting $u = \mathrm{e}^{-\beta\omega_0 N}\nabla \mathrm{e}^{\beta\omega_0 N}\rho$ and using formula (4.6), we can rewrite the eigenvalue equation as

$$\nabla^* D(N)u = 0.$$

Since ∇^* is clearly injective, this means $D(N)u = 0$ and hence $u_n = 0$ unless $D(n) = 0$, that is n is a Rabi resonance. At this stage, we have to distinguish 3 cases.

4.5.1 The Non-resonant Case

If the system is non-resonant, it follows from (4.7) that $D(n) = 0$ if and only if $n = 0$ and hence our eigenvalue equation reduces to

$$u_n = \rho_n - \mathrm{e}^{-\beta\omega_0}\rho_{n-1} = 0,$$

for $n \geq 1$. We conclude that there is a unique diagonal invariant state

$$\frac{\mathrm{e}^{-\beta\omega_0 N}}{\mathrm{Tr}\,\mathrm{e}^{-\beta\omega_0 N}} = \rho_{\mathcal{S}}^{\beta *} = \rho_{\mathcal{S}}^{(1)\,\beta *},$$

if $\beta > 0$ and none if $\beta \leq 0$.

4.5.2 The Simply Resonant Case

If the system is simply resonant there exists $n_1 \in \mathbb{N}^*$ such that $D(n) = 0$ if and only if $n = 0$ or $n = n_1$. The eigenvalue equation then splits into two decoupled systems

$$\rho_n = \mathrm{e}^{-\beta\omega_0}\rho_{n-1}, \quad n \in I_1 \equiv \{1, \ldots, n_1 - 1\},$$
$$\rho_n = \mathrm{e}^{-\beta\omega_0}\rho_{n-1}, \quad n \in I_2 \equiv \{n_1 + 1, \ldots\}.$$

The first one yields the invariant state

$$\frac{\mathrm{e}^{-\beta\omega_0 N} P_1}{\mathrm{Tr}\,\mathrm{e}^{-\beta\omega_0 N} P_1} = \rho_{\mathcal{S}}^{(1)\,\beta *},$$

for any $\beta \in \mathbb{R}$. The second system gives another invariant state

$$\frac{\mathrm{e}^{-\beta\omega_0 N} P_2}{\mathrm{Tr}\,\mathrm{e}^{-\beta\omega_0 N} P_2} = \rho_{\mathcal{S}}^{(2)\,\beta *},$$

provided $\beta > 0$.

4.5.3 The Fully Resonant Case

If the system is fully resonant $D(n)$ has an infinite sequence $n_0 = 0 < n_1 < n_2 < \cdots$ of zeros. The eigenvalue equation now splits into an infinite number of finite dimensional systems

$$\rho_n = \mathrm{e}^{-\beta\omega_0}\rho_{n-1}, \quad n \in I_k \equiv \{n_{k-1} + 1, \ldots, n_k - 1\},$$

where $k = 1, 2, \ldots$. For any $\beta \in \mathbb{R}$, we thus have an infinite number of invariant states

$$\frac{\mathrm{e}^{-\beta\omega_0 N} P_k}{\operatorname{Tr} \mathrm{e}^{-\beta\omega_0 N} P_k} = \rho_{\mathcal{S}}^{(k)\,\beta*},$$

one for each Rabi sector.

4.5.4 Metastable States

If the system is non-resonant we say that $m \in \mathbb{N}^*$ is a Rabi quasi-resonance if it satisfies $D(m) < D(m \pm 1)$. Let $(m_k)_{k\in\mathbb{N}^*}$ be the strictly increasing sequence of quasi-resonances. It is straightforward to show that $D(m_k) = O(k^{-2})$ as $k \to \infty$. This implies that for large k the "quasi Rabi sectors" $\ell^2(\{m_k, \ldots, m_{k+1} - 1\})$ are very weakly coupled. To make this statement more precise let

$$D_0(n) \equiv \begin{cases} 0 & \text{if } n \in \{m_1, m_2, \ldots\}, \\ D(n) & \text{otherwise,} \end{cases}$$

and $\mathcal{L}_{\beta,0}^{(0)} \equiv I - \nabla^* D_0(N) \mathrm{e}^{-\beta\omega_0 N} \nabla \mathrm{e}^{\beta\omega_0 N}$. One immediately concludes that

$$\mathcal{L}_{\beta}^{(0)} = \mathcal{L}_{\beta,0}^{(0)} + \mathcal{T}, \tag{4.8}$$

where $\mathcal{T}$ is a trace class operator. The above analysis of the fully resonant case shows that 1 is an infinitely degenerate eigenvalue of $\mathcal{L}_{\beta,0}^{(0)}$. The corresponding positive eigenvectors

$$\tilde{\rho}_{\mathcal{S}}^{(k)\,\beta*} = \frac{\mathrm{e}^{-\beta\omega_0 N} \widetilde{P}_k}{\operatorname{Tr} \mathrm{e}^{-\beta\omega_0 N} \widetilde{P}_k}$$

where $\widetilde{P}_k$ denotes the orthogonal projection onto $\ell^2(\{0, \ldots, m_k - 1\})$, are metastable states of the system. Because of these almost invariant states, the global relaxation process is extremely slow in the non-resonant and simply resonant cases. In spectral terms, (4.8) shows that 1 is always in the essential spectrum of $\mathcal{L}_\beta$. It follows that relaxation can not be exponential in infinite dimensional Rabi sectors.

As an illustration, we have computed the evolution of the first metastable state $\tilde{\rho}_{\mathcal{S}}^{(1)\,\beta*}$ and the relative entropies

$$\mathfrak{D}_k(n) \equiv -\mathrm{Ent}\big(\mathcal{L}_\beta^n(\tilde{\rho}_{\mathcal{S}}^{(1)\,\beta*}) \,|\, \tilde{\rho}_{\mathcal{S}}^{(k)\,\beta*}\big),$$

in a typical, non-resonant one-atom maser situation (as described in [54]) with atoms in equilibrium at room temperature. We recall that the entropy of a state μ relative to the state ν is defined by

$$\mathrm{Ent}(\mu \,|\, \nu) = \operatorname{Tr} \mu(\log \mu - \log \nu).$$

It is a measure of the "distance" between μ and ν and is also called Kullback–Leibler divergence in information theory. Its main property is $\mathrm{Ent}(\mu \,|\, \nu) \le 0$ where the equality holds iff $\mu = \nu$. Figure 1 shows $\mathfrak{D}_k(n)$ as a function of n for $k = 2, 3, \ldots$ on a log-log scale. It clearly describes the cascade of $\mathcal{L}_\beta^n(\tilde{\rho}_{\mathcal{S}}^{(1)\,\beta*})$ through the sequence of metastable states $\tilde{\rho}_{\mathcal{S}}^{(2)\,\beta*} \to \tilde{\rho}_{\mathcal{S}}^{(3)\,\beta*} \to \cdots$.

Another way to see metastable states in action consists in cooling the cavity with cold atoms. Figure 2 shows the result of such a calculation. The solid line is the initial state of the

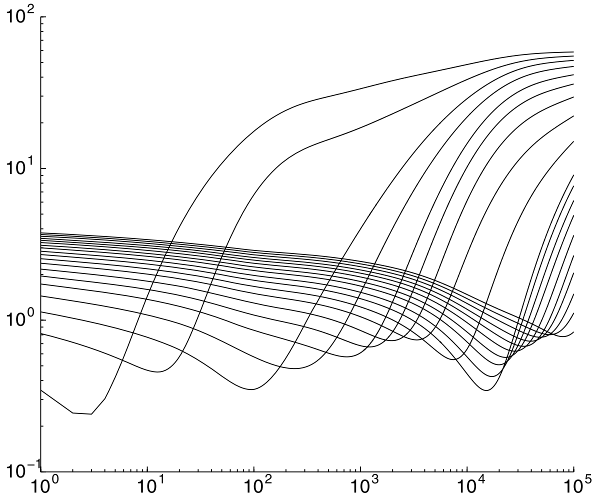

Fig. 1 The metastable cascade (notice the log-log scale!)

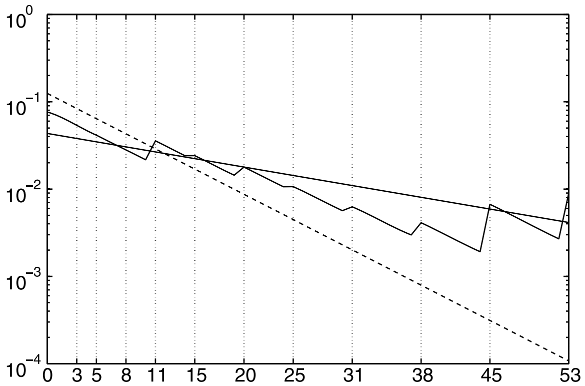

Fig. 2 Cooling the cavity: 5000 interactions

cavity which we chose to be thermal equilibrium with an average photon number of 22. The dashed line is the stationary state $\rho_{\mathcal{S}}^{\beta^*}$, thermal equilibrium with an average of 7 photons. The broken line is the state of the cavity after 5000 interactions. The vertical dashed lines mark the positions of the Rabi quasi-resonances m_k. The picture shows clearly that local equilibrium is achieved in each interval $[m_k, m_{k+1}[$: the slope of the broken line agrees with that of the invariant state on these intervals. However only the first three intervals have reached a common equilibrium. The average photon number at this stage is still slightly larger than 17. It requires 50000 interactions for this number to drop under 10. Figure 3 shows the corresponding state of the cavity. A gross picture of the relaxation process is provided by Fig. 4 where the average photon number is plotted against the number of interactions.

4.6 Rabi Resonances and the Block Structure of $\mathcal{L}_\beta$

To understand the RI dynamics of Rabi-resonant systems we need to investigate the block structure of the map $\mathcal{L}_\beta$ in the presence of r such resonances $n_1, \ldots$. The decomposition

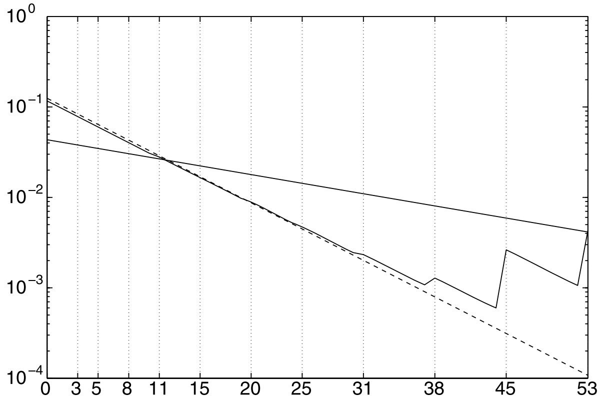

Fig. 3 Cooling the cavity: 50000 interactions

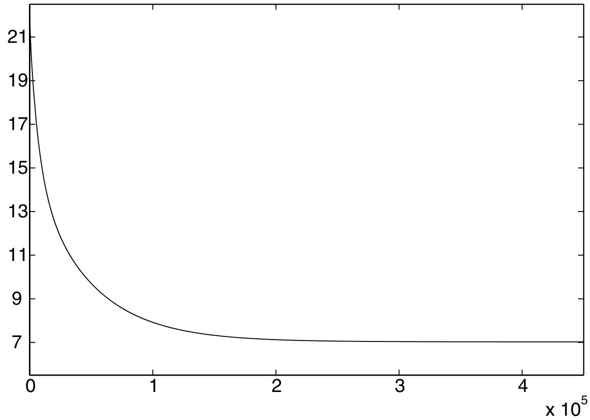

Fig. 4 Cooling the cavity: average photon number

(3.2) of $\mathcal{H}_\mathcal{S}$ into Rabi sectors induces a decomposition

$$\mathcal{J}_1(\mathcal{H}_\mathcal{S}) = \bigoplus_{k,p=1}^{r} \mathcal{J}_1^{(k,p)}(\mathcal{H}_\mathcal{S}), \quad \mathcal{J}_1^{(k,p)}(\mathcal{H}_\mathcal{S}) = P_k \mathcal{J}_1(\mathcal{H}_\mathcal{S}) P_p = \mathcal{J}_1(\mathcal{H}_\mathcal{S}^{(p)}, \mathcal{H}_\mathcal{S}^{(k)}), \tag{4.9}$$

where each term itself decomposes into

$$\mathcal{J}_1^{(k,p)}(\mathcal{H}_\mathcal{S}) = \bigoplus_{d=n_p-n_{k+1}+1}^{n_{p+1}-n_k-1} \mathcal{J}_1^{(k,p,d)}(\mathcal{H}_\mathcal{S}), \tag{4.10}$$

with

$$\mathcal{J}_1^{(k,p,d)}(\mathcal{H}_\mathcal{S}) \equiv \{X \in \mathcal{J}_1^{(k,p)}(\mathcal{H}_\mathcal{S}) \mid \mathrm{e}^{-i\theta N} X \mathrm{e}^{i\theta N} = \mathrm{e}^{i\theta d} X \text{ for all } \theta \in \mathbb{R}\}.$$

It easily follows from the fact that $S(n) = 0$ for $n \in R(\eta, \xi)$ that

$$V_{\sigma'\sigma} P_k = P_k V_{\sigma'\sigma} P_k = P_k V_{\sigma'\sigma}, \qquad V_{\sigma'\sigma}^* P_k = P_k V_{\sigma'\sigma}^* P_k = P_k V_{\sigma'\sigma}^*,$$

hold for any σ, σ' and any Rabi projection P_k. Therefore, one has

$$P_k \mathcal{L}_\beta(\rho) P_p = \mathcal{L}_\beta(P_k \rho P_p),$$

i.e., the map $\mathcal{L}_\beta$ further decomposes into

$$\mathcal{L}_\beta = \bigoplus_{k,p=1}^{r} \mathcal{L}_\beta^{(k,p)}, \quad \mathcal{L}_\beta^{(k,p)} = \bigoplus_{d=n_p-n_{k+1}+1}^{n_{p+1}-n_k-1} \mathcal{L}_\beta^{(k,p,d)}, \tag{4.11}$$

where $\mathcal{L}_\beta^{(k,p,d)}$ is the restriction of $\mathcal{L}_\beta$ to the subspace $\mathcal{J}_1^{(k,p,d)}(\mathcal{H}_\mathcal{S})$. It will be useful to visualize the elements of this subspace as $l_k \times l_p$ matrices (with respect to the canonical basis of $\mathcal{H}_\mathcal{S}$) of the form

$$X = \begin{pmatrix} 0 & \cdots & 0 & x_1 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & x_2 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & x_3 & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Recall that l_n is the dimension of the n-th Rabi sector.

4.7 The Peripheral Point Spectrum of $\mathcal{L}_\beta$

We have obtained all the diagonal eigenvectors to the eigenvalue 1 of $\mathcal{L}_\beta$ in the Sect. 4.5. In this subsection we further investigate the peripheral spectrum of $\mathcal{L}_\beta$, more precisely the eigenvalue problem

$$\mathcal{L}_\beta(X) = \mathrm{e}^{i\theta} X, \tag{4.12}$$

with $\theta \in \mathbb{R}$. The following lemma shows that in almost all cases the only peripheral eigenvalue is 1 and that all the corresponding eigenvectors are diagonal. In other words, there are no solutions to (4.12) except for multiples of those obtained in Sect. 4.5.

Lemma 4.6 1. *The only peripheral eigenvalue of* $\mathcal{L}_\beta^{(0)}$ *is* 1.

2. *If the system is not degenerate, then the only peripheral eigenvalue of* $\mathcal{L}_\beta$ *is* 1 *and the corresponding eigenvectors are diagonal.*

3. *If the system is degenerate we denote* $N(\eta,\xi) \equiv \{n \in \{0\} \cup R(\eta,\xi) \,|\, n+1 \in R(\eta,\xi)\}$ *and* $\mathcal{D}(\eta,\xi) \equiv \{d = n-m \,|\, n,m \in N(\eta,\xi), n \neq m\}$. *In this case the set of peripheral eigenvalues of* $\mathcal{L}_\beta$ *is given by*

$$\{1\} \cup \{\mathrm{e}^{i(\tau\omega+\xi\pi)d} \,|\, d \in \mathcal{D}(\eta,\xi)\}.$$

More precisely, for any $k, p \in \mathbb{N}^*$ *such that* $k \neq p$ *one has*:

- (i) 1 *is the only peripheral eigenvalue of* $\mathcal{L}_\beta^{(k,k)}$ *and the corresponding eigenvectors are diagonal.*
- (ii) $\mathcal{L}_\beta^{(k,p)}$ *has no peripheral eigenvalue except if* n_k *and* n_p *both belong to* $N(\eta,\xi)$ *in which case it has the unique and simple eigenvalue* $\mathrm{e}^{i(\tau\omega+\xi\pi)d}$ *where* $d = n_p - n_k$.

Proof According to the decomposition (4.11) it suffices to consider $X \in \mathcal{J}_1^{(k,p,d)}(\mathcal{H}_\mathcal{S})$ satisfying (4.12). We note that $X^* \in \mathcal{J}_1^{(p,k,-d)}(\mathcal{H}_\mathcal{S})$ then satisfies $\mathcal{L}_\beta(X^*) = \mathrm{e}^{-i\theta} X^*$. It follows from Theorem 4.3 that $Y = (X^*X)^{1/2} \in \mathcal{J}_1^{(p,p,0)}(\mathcal{H}_\mathcal{S})$ as well as $Z = (XX^*)^{1/2} \in \mathcal{J}_1^{(k,k,0)}(\mathcal{H}_\mathcal{S})$ are positive diagonal eigenvectors of $\mathcal{L}_\beta$ to the eigenvalue 1.

If $\beta \le 0$ and $l_k = \infty$ (respectively $l_p = \infty$) it follows from Sect. 4.5 that $Z = 0$ (respectively $Y = 0$) and hence $X = 0$. In the remaining cases on has $Y = \lambda \rho_S^{(p)\,\beta^*}$ and $Z = \mu \rho_S^{(k)\,\beta^*}$ for some $\lambda, \mu \ge 0$. We consider four cases.

Case I: $l_k \ne l_p$ (X is not a square matrix). Without loss of generality (interchanging X and X^*) we may assume that $l_k > l_p$ and in particular that l_p is finite. Then Z is a diagonal $l_k \times l_k$ matrix whose rank does not exceed l_p. It follows that at least one of its diagonal entry is zero. Since $\rho_S^{(k)\,\beta^*} > 0$ we conclude that $\mu = 0$ and hence $X = 0$.

Case II: $l_k = l_p$ and $d \ne n_p - n_k$ (X is square but not diagonal). In this case we can assume (again by interchanging X and X^*) that $d > n_p - n_k$. Then the kernel of X is non-trivial and we can apply the same argument than in case I.

Case III: $l_k = l_p > 1$ and $d = n_p - n_k$ (X is diagonal). In this case we can assume $d \ge 0$. The diagonal elements of X can be written as

$$x_n = \mu\, \mathrm{e}^{i\varphi_n - \beta\omega_0 n}, \quad n \in \{n_k, \dots, n_{k+1} - 1\},$$

for some $\mu \in \mathbb{C}$ and $\varphi_j \in \mathbb{R}$. Assuming $\mu \ne 0$ and using the Kraus representation (4.3), (4.4), the eigenvalue equation (4.12) writes

$$\begin{aligned} &\frac{\mathrm{e}^{i\tau\omega d}}{1 + \mathrm{e}^{-\beta\omega_0}} \Big[\big(a_n \overline{a_{n+d}} + \mathrm{e}^{-\beta\omega_0} \overline{a_{n+1}} a_{n+d+1} \big)\, \mathrm{e}^{i\varphi_n} \\ &\quad + b_n \overline{b_{n+d}} \mathrm{e}^{i\varphi_{n-1}} + \mathrm{e}^{-\beta\omega_0} \overline{b_{n+1}} b_{n+d+1} \mathrm{e}^{i\varphi_{n+1}} \Big] = \mathrm{e}^{i(\theta+\varphi_n)}, \end{aligned} \tag{4.13}$$

for $n \in \{n_k, \dots, n_{k+1} - 1\}$ where

$$a_n \equiv C(n), \qquad b_n \equiv \sqrt{n}\, S(n).$$

One easily checks that $|a_n|^2 + |b_n|^2 = 1$. The resonance condition at n_k and $n_p = n_k + d$ is $b_{n_k} = b_{n_p} = 0$ and hence $|a_{n_k}| = |a_{n_p}| = 1$. Setting $z \equiv \mathrm{e}^{\beta\omega_0}$ and $\alpha \equiv \tau\omega d - \theta$ we can recast (4.13) as

$$z(A_n - 1) = 1 - B_n, \tag{4.14}$$

where

$$\begin{aligned} A_n &= \mathrm{e}^{i\alpha} a_n \overline{a_{n+d}} + \mathrm{e}^{i\alpha - i(\varphi_n - \varphi_{n-1})} b_n \overline{b_{n+d}}, \\ B_n &= \mathrm{e}^{i\alpha} a_{n+d+1} \overline{a_{n+1}} + \mathrm{e}^{i\alpha + i(\varphi_{n+1} - \varphi_n)} b_{n+d+1} \overline{b_{n+1}}. \end{aligned}$$

The Cauchy-Schwarz inequality yields $\operatorname{Re} A_n \le |A_n| \le 1$, $\operatorname{Re} B_n \le |B_n| \le 1$ and hence

$$\operatorname{Re} z(A_n - 1) \le 0, \qquad \operatorname{Re}(1 - B_n) \ge 0.$$

It follows that (4.14) is equivalent to $A_n = B_n = 1$. In order for equality to hold in the Cauchy-Schwarz inequality $\operatorname{Re} A_n \le 1$, we must have

$$a_{n+d} = \mathrm{e}^{i\alpha} a_n, \qquad b_{n+d} = \mathrm{e}^{i\alpha - i(\varphi_n - \varphi_{n-1})} b_n. \tag{4.15}$$

Similarly, to get equality in the inequality $\operatorname{Re} B_n \le 1$ requires

$$a_{n+d+1} = \mathrm{e}^{-i\alpha} a_{n+1}, \qquad b_{n+d+1} = \mathrm{e}^{-i\alpha - i(\varphi_{n+1} - \varphi_n)} b_{n+1}. \tag{4.16}$$

If $d=0$ the first equation in (4.15) and the fact that $a_{n_k}\neq 0$ imply $\mathrm{e}^{i\alpha}=1$. Hence $\mathrm{e}^{i\theta}=\mathrm{e}^{i\tau\omega d}=1$ and X is a multiple of the invariant state $\rho_{\mathcal{S}}^{(k)\beta^*}$. We can therefore assume that $d>0$ and hence $n_p>0$. Since $b_{n_k+1}\neq 0$ and $b_{n_p+1}\neq 0$, comparing the second equations in (4.15) at $n=n_k+1$ and (4.16) at $n=n_k$ allows us to conclude that $\mathrm{e}^{i\alpha}$ is real.

We shall now consider separately the two cases $\eta=0$ and $\eta\neq 0$. In the first case, the first equation in (4.16) implies

$$\cos^2\pi\sqrt{\xi(n_p+1)}=\cos^2\pi\sqrt{\xi(n_k+1)}$$

and therefore

$$\sqrt{\xi(n_p+1)}+\varepsilon\sqrt{\xi(n_k+1)}=r, \tag{4.17}$$

for some $\varepsilon\in\{\pm 1\}$ and some integer $r>0$. Using the resonance condition

$$\xi n_p=q^2,$$

for some integer $q>0$, we can rewrite (4.17) as

$$\varepsilon\sqrt{\frac{n_k+1}{n_p}}=\frac{r}{q}-\sqrt{\frac{n_p+1}{n_p}}.$$

Squaring both sides of this equality leads to

$$\frac{n_k+1}{n_p}=\frac{r^2}{q^2}+\frac{n_p+1}{n_p}-\frac{2r}{q}\sqrt{\frac{n_p+1}{n_p}},$$

which leads to a contradiction since the square root on the right hand side of the last equality is always irrational.

If $\eta\neq 0$, rewriting the imaginary part of the first equation in (4.15) as

$$\eta^{1/2}\frac{\sin\pi\sqrt{\xi(n+d)+\eta}}{\sqrt{\xi(n+d)+\eta}}=\pm\eta^{1/2}\frac{\sin\pi\sqrt{\xi n+\eta}}{\sqrt{\xi n+\eta}},$$

and comparing it with the second equation in (4.15)

$$\sqrt{\xi(n+d)}\frac{\sin\pi\sqrt{\xi(n+d)+\eta}}{\sqrt{\xi(n+d)+\eta}}=\pm\mathrm{e}^{-i(\varphi_n-\varphi_{n-1})}\sqrt{\xi n}\frac{\sin\pi\sqrt{\xi n+\eta}}{\sqrt{\xi n+\eta}},$$

we get $\sqrt{\xi(n+d)}=\mathrm{e}^{-i(\varphi_n-\varphi_{n-1})}\sqrt{\xi n}$ which contradicts our hypothesis $d>0$.

Case IV: $l_k=l_p=1$ and $d=n_p-n_k$ (X is scalar). We follow the same argument as in case III. Now the second equations in (4.15), (4.16) are trivially satisfied and only the two equations

$$a_{n_p}=\mathrm{e}^{i\alpha}a_{n_k},\qquad a_{n_p+1}=\mathrm{e}^{-i\alpha}a_{n_k+1} \tag{4.18}$$

survive. In the case $d=0$ one can conclude, as in case III, that $\mathrm{e}^{i\theta}=1$. We can therefore assume that $d>0$ and $n_p>0$, which means that (n_k,n_k+1), (n_p,n_p+1) are two distinct pairs of consecutive resonances, i.e., that the system is degenerate. In this case, (4.18) are easily seen to be satisfied with $\mathrm{e}^{i\theta}=(-1)^{\xi d}\mathrm{e}^{i\tau\omega d}$. □

Remark Note that $N(\eta,\xi)$ is a finite set. Indeed, if $n \in N(\eta,\xi)$ there exist positive integers p and q such that $\xi n + \eta = p^2$ and $\xi(n+1)+\eta = q^2$. Hence, $\xi = q^2 - p^2 = (q-p)(q+p)$ and therefore $p \le p+q \le \xi$ so that $n \le \frac{\xi^2-\eta}{\xi}$. As a consequence $\mathcal{D}(\eta,\xi)$ is also a finite set as we mentioned in Sect. 3. There is some numerical evidence that $N(\eta,\xi)$ contains at most two elements, but an analytic proof of this conjecture seems very difficult.

4.8 Ergodicity and Relaxation

4.8.1 Proof of Theorem 3.3

It is now easy to prove that the diagonal invariant states obtained in Sect. 4.5 are ergodic. Each such state is of the form $\rho = \rho_{\mathcal{S}}^{(k)\,\beta^*}$ for some k and hence its support is a Rabi projection P_k. Any state μ such that $\mu \ll \rho$ is an element of $\mathcal{J}_1^{(k,k)}(\mathcal{H}_{\mathcal{S}}) = \mathcal{J}_1(\mathcal{H}_{\mathcal{S}}^{(k)})$. In particular $\mathcal{L}_\beta(\mu) = \mathcal{L}_\beta^{(k,k)}(\mu)$ and it is therefore sufficient to prove ergodicity of ρ with respect to the semigroup generated by $\mathcal{L}_\beta^{(k,k)}$. Lemma 4.6 implies that $\rho_{\mathcal{S}}^{(k)\,\beta^*}$ is the unique faithful invariant state for this semigroup. Ergodicity follows from Theorem 4.4.

1. In the non-resonant case the unique ergodic state $\rho_{\mathcal{S}}^{(1)\,\beta^*} = \rho_{\mathcal{S}}^{\beta^*}$ is faithful and hence one has

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} \left(\mathcal{L}_\beta^n(\mu)\right)(A) = \rho_{\mathcal{S}}^{\beta^*},$$

for all states μ and all $A \in \mathcal{B}(\mathcal{H}_{\mathcal{S}})$.

2. In the simply resonant cases we shall first consider initial states $\mu \in \oplus_{|k|\le d}\, \mathcal{J}_1^{(k)}(\mathcal{H}_{\mathcal{S}})$ for finite $d \in \mathbb{N}$. According to (4.9), (4.10), such a state can be decomposed into a finite sum

$$\mu = \mu^{(1,1)} \oplus \mu^{(2,2)} \oplus \left(\bigoplus_{j=1}^{d} \mu^{(1,2,j)}\right) \oplus \left(\bigoplus_{j=-d}^{-1} \mu^{(2,1,j)}\right)$$

and hence

$$\mathcal{L}_\beta^n(\mu) = \mathcal{L}_\beta^{(1,1)\,n}(\mu^{(1,1)}) \oplus \mathcal{L}_\beta^{(2,2)\,n}(\mu^{(2,2)}) \oplus \left(\bigoplus_{j=1}^{d} \mathcal{L}_\beta^{(1,2,j)\,n}(\mu^{(1,2,j)})\right) \oplus \left(\bigoplus_{j=-d}^{-1} \mathcal{L}_\beta^{(2,1,j)\,n}(\mu^{(2,1,j)})\right).$$

Since the operators $\mathcal{L}_\beta^{(1,2,j)}$ and $\mathcal{L}_\beta^{(2,1,j)}$ act on finite dimensional spaces they have a finite number of eigenvalues which, by Lemma 4.6, all lie strictly inside the unit disk. It follows that the corresponding terms in the above sum decay (exponentially) as $n \to \infty$. The first two terms in this sum can be handled as in the non-resonant case since the two Rabi sectors $\mathcal{H}_{\mathcal{S}}^{(1)}$ and $\mathcal{H}_{\mathcal{S}}^{(2)}$ are equipped with unique faithful invariant states $\rho_{\mathcal{S}}^{(1)\,\beta^*}$ and $\rho_{\mathcal{S}}^{(2)\,\beta^*}$. Therefore, for any $A \in \mathcal{B}(\mathcal{H}_{\mathcal{S}})$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} \left(\mathcal{L}_\beta^n(\mu)\right)(A) = \mu^{(1,1)}(I)\,\rho_{\mathcal{S}}^{(1)\,\beta^*}(A) + \mu^{(2,2)}(I)\,\rho_{\mathcal{S}}^{(2)\,\beta^*}(A), \tag{4.19}$$

and (3.5) follows from the fact that $\mu^{(k,k)}(I) = \mu(P_k)$. On the left hand side of (4.19) the Cesàro mean is uniformly continuous in μ (with respect to N) while the right hand side is continuous. Equation (4.19) therefore extends by continuity to any state μ in the closure of $\bigcup_{d\in\mathbb{N}}(\bigoplus_{|k|\le d} \mathcal{J}_1^{(k)}(\mathcal{H}_\mathcal{S}))$. The next lemma shows that this is all of $\mathcal{J}_1(\mathcal{H}_\mathcal{S})$.

Lemma 4.7 *For any state μ there exists a sequence $(\mu_k)_{k\in\mathbb{N}}$ in $\mathcal{J}_{1+}(\mathcal{H}_\mathcal{S})$ such that*

$$\mu_k \in \bigoplus_{|d|\le k} \mathcal{J}_1^{(d)}(\mathcal{H}_\mathcal{S})$$

and $\lim_{k\to\infty} \mu_k = \mu$ *in* $\mathcal{J}_1(\mathcal{H}_\mathcal{S})$.

Proof We first note that $\theta \mapsto \mu(\theta) \equiv \mathrm{e}^{-i\theta N}\mu\mathrm{e}^{i\theta N}$ is a continuous, 2π-periodic function from $\mathbb{R}$ to $\mathcal{J}_{1+}(\mathcal{H}_\mathcal{S})$ with Fourier coefficients

$$\mu^{(d)} \equiv \int_0^{2\pi} \mu(\theta)\,\mathrm{e}^{-i\theta d}\,\frac{\mathrm{d}\theta}{2\pi}.$$

By (2.5), one has $\mu^{(d)} \in \mathcal{J}_1^{(d)}(\mathcal{H}_\mathcal{S})$ and hence

$$\mu_{k-1} \equiv \frac{1}{k}\sum_{j=0}^{k-1}\left(\sum_{d=-j}^{j} \mu^{(d)}\mathrm{e}^{i\theta d}\right) \in \bigoplus_{|d|\le k-1} \mathcal{J}_1^{(d)}(\mathcal{H}_\mathcal{S}).$$

By Fejér's integral formula (see e.g., [51])

$$\mu_{k-1} = \int_0^{\pi} F_k(\theta)(\mu(\theta)+\mu(-\theta))\,\mathrm{d}\theta,$$

where

$$F_k(\theta) \equiv \frac{1}{2\pi k}\,\frac{\sin^2(k\theta/2)}{\sin^2(\theta/2)},$$

is Fejér's kernel. Since $F_k \ge 0$, it follows that $\mu_k \ge 0$. Finally, from

$$\mu_k - \mu = \int_0^{\pi} F_k(\theta)(\mu(\theta)+\mu(-\theta)-2\mu)\,\mathrm{d}\theta,$$

we obtain the estimate

$$\|\mu_k - \mu\|_1 \le \int_0^{\pi} F_k(\theta)\|\mu(\theta)+\mu(-\theta)-2\mu\|_1\,\mathrm{d}\theta,$$

whose right hand side vanishes as $k \to \infty$ by Fejér's convergence theorem (see the proof of Theorem 13.32 in [51]). □

3. In the fully resonant, non-degenerate case we start with an arbitrary state μ and introduce a cutoff by means of the orthogonal projections

$$P_{\le n} \equiv \sum_{j=1}^{n} P_j.$$

Setting $\mu_{\le n} \equiv P_{\le n}\mu P_{\le n}$, using the decomposition into a finite sum of finite dimensional blocks

$$\mu_{\le n} = \bigoplus_{k,p=1}^{n} \left(\bigoplus_{d=n_p-n_{k+1}+1}^{n_{p+1}-n_k-1} \mu^{(k,p,d)} \right),$$

and proceeding as in the simply resonant case we obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} \left(\mathcal{L}_\beta^n(\mu_{\le n})\right)(A) = \sum_{j=1}^{n} \mu^{(j,j)}(I)\, \rho_{\mathcal{S}}^{(j)\,\beta^*}(A). \tag{4.20}$$

Since $\lim_{n\to\infty} \mu_{\le n} = \mu$ in $\mathcal{J}_1(\mathcal{H}_{\mathcal{S}})$ and $\sum_{j=1}^{\infty} \mu^{(j,j)}(I) = \mu(I) = 1$, (4.20) extends to μ, which proves (3.6).

4. The last assertion of Theorem 3.3 is a direct consequence of Lemma 4.6.

4.8.2 Proof of Theorem 3.4

When $\mathcal{H}_{\mathcal{S}}^{(k)}$ is finite dimensional, one can say more. By Lemma 4.6 the spectrum of $\mathcal{L}_\beta^{(k,k)}$ consists in a simple eigenvalue 1 with eigenvector $\rho_{\mathcal{S}}^{(k)\,\beta^*}$ and finitely many eigenvalues located in a disk $\{z \in \mathbb{C} \,|\, |z| \le R\}$ of radius $R < 1$. This implies that

$$\|\mathcal{L}_\beta^n(\mu) - \rho_{\mathcal{S}}^{(k)\,\beta^*}\|_1 \le C_k \mathrm{e}^{-\alpha_k n},$$

for some positive constants C_k, α_k and all state $\mu \ll \rho_{\mathcal{S}}^{(k)\,\beta^*}$. Thus $\rho_{\mathcal{S}}^{(k)\,\beta^*}$ is (exponentially) mixing.

4.9 Proof of Theorem 4.4

Theorem 4.4 resembles the von Neumann mean ergodic theorem. However, the latter holds in full generality only for contractions on *reflexive* Banach spaces, which is not the case of $\mathcal{J}_1(\mathcal{H})$. To bypass this problem, we shall work in a Hilbert space representation.

Let $\mathfrak{M} = \mathcal{B}(\mathcal{H})$ denote the von Neumann algebra of observables on $\mathcal{H}$ and $(\mathcal{K}, \pi, \Psi)$ be the GNS representation of $\mathfrak{M}$ associated to the invariant state ρ_{stat} (see e.g., [12]). On the dense subspace $\mathcal{K}_0 \equiv \pi(\mathfrak{M})\Psi \subset \mathcal{K}$ we define the map

$$M : \pi(A)\Psi \mapsto \pi(\phi^*(A))\Psi, \tag{4.21}$$

where ϕ^* acts on $\mathfrak{M}$ and is the dual map of ϕ. The operator M implements the map ϕ^* in the GNS representation. The following lemma is rather general. It actually holds as soon as the initial map satisfies the Kadison-Schwarz inequality (4.22) (e.g. if it is a 2-positive map) and the reference state is invariant [1].

Lemma 4.8 *M extends to a contraction on $\mathcal{K}$.*

Proof The map ϕ^* is a completely positive map. Hence it satisfies the Kadison-Schwarz inequality (see e.g. [39])

$$\phi^*(A^*A) \ge \phi^*(A)^*\phi^*(A), \tag{4.22}$$

for all $A \in \mathcal{B}(\mathcal{H})$. In particular we have, for any $A \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned}\|M\pi(A)\Psi\|^2 &= \langle\Psi|\pi(\phi^*(A)^*\phi^*(A))\Psi\rangle \\ &= \rho_{\rm stat}\left(\phi^*(A)^*\phi^*(A)\right) \\ &\leq \rho_{\rm stat}(\phi^*(A^*A)) \\ &= \rho_{\rm stat}(A^*A) \\ &= \|\pi(A)\Psi\|^2,\end{aligned}$$

where we have used that $\rho_{\rm stat}$ is an invariant state to get the 4th line. The operator M thus defines a contraction on $\mathcal{K}_0$ and hence extends to a contraction on $\mathcal{K}$. □

Let ρ be any normal state. Then there exists $\Phi \in \mathcal{K}$ such that $\rho(A) = \langle\Phi|\pi(A)\Phi\rangle$ (see e.g. [12, 45]). It is therefore sufficient to prove that for any normalized vector $\Phi \in \mathcal{K}$, and any observable $A \in \mathfrak{M}$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle\Phi|\pi(\phi^{*n}(A))\Phi\rangle = \langle\Psi|\pi(A)\Psi\rangle. \tag{4.23}$$

Moreover, since $\rho_{\rm stat}$ is faithful, the vector Ψ is also cyclic for the commutant algebra $\pi(\mathfrak{M})'$. We may therefore prove (4.23) only for vectors of the form $\Phi = B'\Psi$ where $B' \in \pi(\mathfrak{M})'$. For such vectors, we have

$$\begin{aligned}\langle\Phi|\pi(\phi^{*n}(A))\Phi\rangle &= \langle B'^*B'\Psi|\pi(\phi^{*n}(A))\Psi\rangle \\ &= \langle B'^*B'\Psi|M^n\pi(A)\Psi\rangle.\end{aligned} \tag{4.24}$$

Since M is a contraction on the Hilbert space $\mathcal{K}$, the von Neumann mean ergodic theorem asserts that

$$\operatorname*{s-lim}_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}M^n = P,$$

where P is the projection onto $\mathrm{Ker}(M-I)$ along $\overline{\mathrm{Ran}(M-I)} = \mathrm{Ker}(M^*-I)^\perp$.

Lemma 4.9 $\mathrm{Ker}(M^*-I) = \mathbb{C}\Psi$.

Proof Clearly, $\Psi \in \mathrm{Ker}(M^*-I)$. Conversely, let $\Phi \in \mathcal{K}$ such that $M^*\Phi = \Phi$. Consider the linear functional $\omega : \mathfrak{M} \ni A \mapsto \langle\Phi|\pi(A)\Psi\rangle \in \mathbb{C}$. It is easy to see that ω is normal on $\mathfrak{M}$. Hence, there exists $X \in \mathcal{J}_1(\mathcal{H})$ such that $\omega(A) = \mathrm{Tr}(XA)$. Moreover, for any $A \in \mathfrak{M}$

$$\begin{aligned}\mathrm{Tr}(XA) &= \langle\Phi|\pi(A)\Psi\rangle \\ &= \langle M^*\Phi|\pi(A)\Psi\rangle \\ &= \langle\Phi|M\pi(A)\Psi\rangle \\ &= \langle\Phi|\pi(\phi^*(A))\Psi\rangle \\ &= \mathrm{Tr}(X\phi^*(A)) \\ &= \mathrm{Tr}(\phi(X)A).\end{aligned}$$

Thus, X is a trace class operator invariant for ϕ. Therefore there exists $\lambda \in \mathbb{C}$ such that $X = \lambda\rho_{\text{stat}}$ and we have for any $A \in \mathfrak{M}$,

$$\langle\Phi|\pi(A)\Psi\rangle = \lambda\langle\Psi|\pi(A)\Psi\rangle.$$

Since Ψ is cyclic for $\pi(\mathfrak{M})$ this proves that $\Phi \in \mathbb{C}\Psi$. □

Using the above lemma, and since $M\Psi = \Psi$, the von Neumann mean ergodic theorem asserts that

$$\operatorname*{s-lim}_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} M^n = |\Psi\rangle\langle\Psi|.$$

Together with (4.24), we get, using the fact that $\Phi = B'\Psi$ is a normalized vector,

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\langle\Phi|\pi(\phi^{*n}(A))\Phi\rangle &= \langle B'^{*}B'\Psi|\Psi\rangle\,\langle\Psi|\pi(A)\Psi\rangle\\ &= \langle\Psi|\pi(A)\Psi\rangle,\end{aligned}$$

which concludes the proof.

4.10 The Resonance Condition

Assertions 1, 2 and 3 of Lemma 3.2 are elementary and their proof is left to the reader. To prove assertion 4 we consider the conditions for consecutive Rabi resonances.

In the perfectly tuned case $\eta = 0$, the only possible consecutive resonances are 0 and 1. Indeed, if $n > 0$ then n and $n+1$ are resonances iff $\xi n = p^2$ and $\xi(n+1) = q^2$ for positive integers p and q. It follows that

$$\sqrt{\frac{n}{n+1}} = \frac{p}{q},$$

which contradicts the irrationality of the square root on the left hand side.

For $\eta > 0$, the conditions for consecutive resonances $0 \le n < n+1 \le m < m+1$ are

$$\begin{aligned} n = 0 \quad \text{or} \quad \xi n + \eta &= p^2, & \xi(n+1) + \eta &= q^2,\\ \xi m + \eta &= p'^2, & \xi(m+1) + \eta &= q'^2,\end{aligned}$$

for positive integers p, p', q, q'. It easily follows that $\xi = q'^2 - p'^2$ and $\eta = p'^2 - \xi m$ from which we conclude that ξ and η must be integers and η a quadratic residue modulo ξ.

Remark Degenerate systems exist, as the following examples show: $N(720, 241) = \{1, 2\}$ and $N(840, 1) = \{1, 52\}$, hence $\mathcal{D}(720, 241) = \{1\}$ and $\mathcal{D}(840, 1) = \{51\}$. Indeed on has

$$720 + 241 = 31^2, \qquad 2\cdot 720 + 241 = 41^2, \qquad 3\cdot 720 + 241 = 49^2,$$

as well as

$$840 + 1 = 29^2, \qquad 2\cdot 840 + 1 = 41^2, \qquad 52\cdot 840 + 1 = 209^2, \qquad 53\cdot 840 + 1 = 211^2.$$

We do not know of an example where $\mathcal{D}(\xi,\eta)$ contains more than one element. Even though it is quite easy to compute $\mathcal{D}(\xi,\eta)$ for given ξ and η, the problem of characterizing the set of integers ξ, η such that $\mathcal{D}(\xi,\eta)$ is non-trivial is extremely difficult. In the special case $m-n=1$ the consecutive resonances condition leads to the Diophantine system

$$\xi = q^2 - p^2 = r^2 - q^2, \qquad \eta = p^2 - n\xi,$$

which contains the subsystem

$$q^2 - \xi = p^2, \qquad q^2 + \xi = r^2,$$

so that p^2, q^2, r^2 are three perfect squares in arithmetic progression. Positive integers ξ for which this system has at least one solution are called *congruent numbers*. The problem of characterizing congruent numbers, the so called *congruum problem*, has a very long history and is still an active research area of number theory (see [52] for a recent breakthrough).

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